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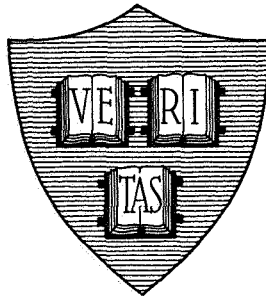
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**PARAMETRIC INTERACTIONS IN A DIELECTRIC MEDIUM
WITH SECOND-HARMONIC SPACE-TIME PUMP MODULATION**

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By

B. Rama Rao



Scientific Report No. 4

August 1969

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Prepared under Grant NGR 22-007-056

Division of Engineering and Applied Physics

Harvard University • Cambridge, Massachusetts

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ABSTRACT

A dispersion relation involving all space-time harmonics has been obtained for electromagnetic wave propagation in an infinite dielectric medium whose permittivity is modulated by a high intensity pump wave containing two harmonics. Brillouin diagrams are given for the parametric coupling between the $n = -1$ (difference frequency) and $n = +1$ (sum frequency) space-time harmonics of the electromagnetic signal wave. The effect of the amplitude and phase of the second harmonic of the pump wave on the parametric growth process has been studied. Both stable and temporally unstable interactions are considered.

Traveling wave parametric interactions of electromagnetic waves propagating in a dielectric medium with permittivity varying periodically in time and space has assumed considerable importance because of its numerous practical applications. Although many theoretical papers [1]-[4] have appeared on this problem, it is assumed for the most part that the perturbing pump wave modulating the medium is a weak, progressive, sinusoidal disturbance. However, there are a number of physical situations in which the modulating pump wave can have a large amplitude and contain two or more harmonics of the fundamental pumping frequency. Stimulated Brillouin scattering processes in Quantum Electronics [5], optical diffraction by non-linear ultrasonic waves [6] or electromagnetic diffraction by large amplitude traveling wave disturbances in the ionosphere or in laboratory plasmas [7] are all various manifestations of this phenomena. The purpose of this report is to study the effect of the amplitude and phase of the second harmonic component of the pump wave on the parametric interactions resulting from electromagnetic wave propagation in dielectric media of this type.

It is assumed that the pump wave traveling along the z direction in an infinite, dielectric medium contains two harmonics and causes the dielectric constant of the medium to vary in the following manner

$$\epsilon(z, t) = \epsilon_0 + \epsilon_1 \cos(\omega_v t - k_v z) + \epsilon_2 \cos(2\omega_v t - 2k_v z + \phi) \quad (1)$$

where $\epsilon_2 \ll \epsilon_1 \ll \epsilon_0$. ω_v and k_v are the angular frequency and wave number of the fundamental component of the pump wave and ϕ is the phase angle of the second harmonic relative to the fundamental. The

pump wave at frequency $2\omega_v$ can either be the second harmonic of a high-intensity, non-linear pump wave in which case the phase angle $\phi = 0$, or an independent wave generated by a separate source in which case the phase angle ϕ can be retarded by various values. For the purpose of this report only the phase angles $\phi = 0$ and $\phi = \pi$ are considered here.

In Equation 1 it is assumed that

$$\epsilon_2 \ll \epsilon_1 \ll \epsilon_0 \quad (2)$$

This assumption is true only if the pump wave has a weak second harmonic component; in most of the physical situations described earlier the second harmonic pump component is indeed small and Equation 2 is applicable [5]-[7].

Let us assume a TEM type electromagnetic wave moving in the z direction and parallel to the pump wave. The wave equation for the electric field $E(z, t)$ in the medium can be written as:

$$\frac{\partial^2 E(z, t)}{\partial z^2} = \mu_0 \frac{\partial^2 D(z, t)}{\partial t^2} \quad (3)$$

μ_0 is the permeability of the medium and the displacement $D = \epsilon(z, t)E$ where $\epsilon(z, t)$ is given by (1). By using Floquet's theorem in conjunction with the principle of superposition, it can be shown that the electric field can be represented in the following form:

$$E(z, t) = E_0 e^{-j(\omega_\ell t - k_\ell z)} \sum_{n=-\infty}^{\infty} A_n e^{-jn(\omega_v t - k_v z)} \quad (4)$$

where ω_ℓ and k_ℓ are the angular frequency and wave number of the signal wave in the unmodulated medium. The phase angle $\phi = 0$ will be considered first and the dispersion relation for the $\phi = \pi$ case will be obtained by

making the necessary change in the sign of the second harmonic modulating coefficient ϵ_2 later in this report. By substituting (4) in (3) we obtain the following fourth-order, linear, homogeneous difference-differential equation; when the phase angle $\phi = 0$:

$$D_n A_n + \frac{\epsilon_1}{2\epsilon_0} [A_{n-1} + A_{n+1}] + \frac{\epsilon_2}{2\epsilon_0} [A_{n-2} + A_{n+2}] = 0 \quad (5)$$

where $n = 0, \pm 1, \pm 2, \dots$ and

$$D_n = \left[1 - c_0^2 \frac{(k_\ell + nk_v)^2}{(\omega_\ell + n\omega_v)^2} \right] \quad (6)$$

$c_0 = 1/\sqrt{\mu_0\epsilon_0}$ is the velocity of the electromagnetic wave in the unmodulated medium. Eq. (5) represents an infinite set of linear difference equations in the modal amplitudes A_n . It will now be shown that solutions can be obtained in the form of rapidly converging continued fractions.

By adding the $(n-1)$ th and $(n+1)$ th equations given by (5) and substituting the resulting expression for $[A_{n-2} + A_{n+2}]$ in (5) we obtain:

$$\begin{aligned} D_n A_n + \frac{\epsilon_1}{2\epsilon_0} [A_{n-1} + A_{n+1}] - \frac{\epsilon_2}{\epsilon_1} [D_{n-1} A_{n-1} + D_{n+1} A_{n+1}] - \frac{\epsilon_2}{\epsilon_0} A_n \\ - \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_0} \right) \left(\frac{\epsilon_2}{\epsilon_1} \right) [A_{n-1} + A_{n+1}] - \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_0} \right) \left(\frac{\epsilon_2}{\epsilon_1} \right) [A_{n-3} + A_{n+3}] \\ = 0 \end{aligned} \quad (7)$$

From Eq. (2) it is seen that $(\epsilon_2/\epsilon_0) \ll 1$ and $(\epsilon_2/\epsilon_1) \ll 1$. Hence, the coefficients of the last two terms in (7) which involve the quadratic products of (ϵ_2/ϵ_0) and (ϵ_2/ϵ_1) are much smaller than the coefficients of the other terms in (7). Therefore, the last two terms in (7) can be

neglected and we obtain a linear, second-order difference-differential equation with rational coefficients:

$$\left[D_n - \frac{\epsilon_2}{\epsilon_0}\right]A_n + \left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n-1}\right]A_{n-1} + \left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n+1}\right]A_{n+1} = 0 \quad (8)$$

The electric field amplitudes can be expressed now in the form of continued fractions as follows:

$$\frac{A_n}{A_{n-1}} = \frac{-\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n-1}\right]}{\left[D_n - \frac{\epsilon_2}{\epsilon_0}\right] - \frac{\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n+1}\right]\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_n\right]}{\left[D_{n+1} - \frac{\epsilon_2}{\epsilon_0}\right] \dots} \quad (9)$$

for $n \geq 1$

and

$$\frac{A_n}{A_{n+1}} = \frac{-\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n+1}\right]}{\left[D_n - \frac{\epsilon_2}{\epsilon_0}\right] - \frac{\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_{n-1}\right]\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1}D_n\right]}{\left[D_{n-1} - \frac{\epsilon_2}{\epsilon_0}\right] \dots} \quad (10)$$

for $n \leq 1$

Equations 9 and 10 apply to the case where $\phi = 0$. The electric field amplitudes when the phase angle $\phi = \pi$ can be obtained by replacing ϵ_2 in the equations 5 to 10 by $-\epsilon_2$.

A dispersion relation for the n^{th} space-time harmonic in the form of continued fractions, for the two cases when $\phi = 0$ and $\phi = \pi$ is given by

$$\begin{aligned}
& \left[D_n \pm \frac{\epsilon_2}{\epsilon_0} \right] - \left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_n \right] \left\{ \frac{\left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n-1} \right]}{\left[D_{n-1} \pm \frac{\epsilon_2}{\epsilon_0} \right] - \left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n-1} \right] \left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n-2} \right]} \right. \\
& \quad \left. \left[D_{n-2} \pm \frac{\epsilon_2}{\epsilon_0} \right] - \dots \right. \\
& \quad + \frac{\left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n+1} \right]}{\left[D_{n+1} \pm \frac{\epsilon_2}{\epsilon_0} \right] - \left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n+1} \right] \left[\frac{\epsilon_1}{2\epsilon_0} \pm \frac{\epsilon_2}{\epsilon_1} D_{n+2} \right]} \left\{ \right. \\
& \quad \left. \left[D_{n+2} \pm \frac{\epsilon_2}{\epsilon_0} \right] - \dots \right\} \quad (11)
\end{aligned}$$

The $\pm \epsilon_2$ in the above equation indicates whether the second harmonic of the pump wave is in phase ($\phi = 0$) or out of phase ($\phi = \pi$) with the fundamental component of the pump wave, respectively. The convergence of the continued fractions in (9), (10) and (11) will now be proved. From (8) and (6) it is seen that the following limit exists:

$$\lim_{n \rightarrow \infty} D_n = \lim_{n \rightarrow \infty} D_{n+1} = \lim_{n \rightarrow \infty} D_{n-1} = \left[1 - \frac{c_0^2}{V^2} \right] = D \quad (12)$$

where $V = \omega_v/k_v$ = the phase velocity of the pump wave.

In the asymptotic limit $n \rightarrow \infty$ the coefficients of the difference-differential equation (8) are constants. Hence, by Laplace's method, it can be now shown that equation (8) is a Poincaré difference equation [3, 8] whose characteristic equation is given by:

$$\rho^2 + \frac{\left[D - \frac{\epsilon_2}{\epsilon_0} \right]}{\left[\frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1} D \right]} + 1 = 0 \quad (13)$$

By applying Poincaré's theorem [3, 8] it can be shown that the finite limits

$$\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = \rho_1 \text{ and } \lim_{n \rightarrow \infty} \frac{A_{n-1}}{A_n} = \rho_2 \text{ exist if}$$

$$\left| D - \frac{\epsilon_2}{\epsilon_0} \right| > 2 \left| \frac{\epsilon_1}{2\epsilon_0} - \frac{\epsilon_2}{\epsilon_1} D \right| . \quad (14)$$

When $(\epsilon_2/\epsilon_1) \ll 1$, the above condition indicates the existence of finite limits except in the sonic region given by

$$\frac{1}{\sqrt{1 + \frac{\epsilon_1}{\epsilon_0} \left[1 + \frac{\epsilon_2}{\epsilon_1} \right]}} \leq v \leq \frac{1}{\sqrt{1 - \frac{\epsilon_1}{\epsilon_0} \left[1 - \frac{\epsilon_2}{\epsilon_1} \right]}} \quad (15)$$

where $v^2 = V^2/c_0^2$. The existence of a solution for (8) automatically assures the convergence of the continued fractions in (9), (10) and (11).

If the second harmonic modulating coefficient ϵ_2 is set equal to zero then eqs. (8)-(11) and (14) reduce to the form given by Hessel and Oliner [9]. In particular, when $\epsilon_2 = 0$, (15) reduces to the form

$$\frac{1}{\sqrt{1 + \frac{\epsilon_1}{\epsilon_0}}} \leq v \leq \frac{1}{\sqrt{1 - \frac{\epsilon_1}{\epsilon_0}}}$$

This condition is identical to the limits of the sonic region defined by Cassedy and Oliner [1] for wave propagation in a sinusoidally modulated dielectric medium.

Outside the sonic region defined in Equation 15, the parametric interactions can be divided into two types as follows: Type 1. When the phase velocity of the pump wave is less than the phase velocity of the signal wave ($v = V/c_0 < 1$), then the parametric interactions are of

the stable, non-inverting modulator type with the dispersion diagram showing stop bands in the phase synchronous region. Stimulated Brillouin scattering and optical diffraction by ultrasound come under this category. Figs. 1a and b show the Brillouin diagram (ω vs. k) for the coupling between $n = -1$ (difference frequency) and $n = +1$ (sum frequency) space-time harmonics. Stokes and Antistokes interactions of this type are of interest in non-linear optics and in ultrasonics which was one of the main reasons for considering them in this report. The parameters chosen in the calculations were $\epsilon_1/\epsilon_0 = 0.5$, $v = V/c_0 = 0.25$, and $\epsilon_2/\epsilon_1 = 0, 0.1$, and -0.1 . The solutions were obtained with an IBM 7094 using a rapidly iterative secant technique for solving the transcendental equation. It can be seen from Fig. 1 that the stop band region expands and contracts depending on the amplitude and phase of the second harmonic pump modulating coefficient ϵ_2 . In the absence of the second pump harmonic the stop band region in terms of $k_0\lambda_v$ extends from 1.974π to 2.098π . When the second harmonic is out of phase with the fundamental component of the pump wave ($\phi = \pi$) and $|\epsilon_2/\epsilon_1| = +0.1$, the width of the stop band increases and extends from 1.953π to 2.123π . When the second harmonic is in phase with the fundamental ($\phi = 0$) and $|\epsilon_2/\epsilon_1| = 0.1$, the stop band width decreases to 1.993π to 2.077π .

Within the stop band, the solutions to the dispersion relation require complex propagation constants indicating contraflow directional coupling resulting in frequency down conversion when the signal and pump waves are traveling in the same direction. For each set of values of ϵ_1 , ϵ_2 , and ϕ the real part of the propagation constant β_L is constant throughout the stop band while the imaginary component α_L varies in a

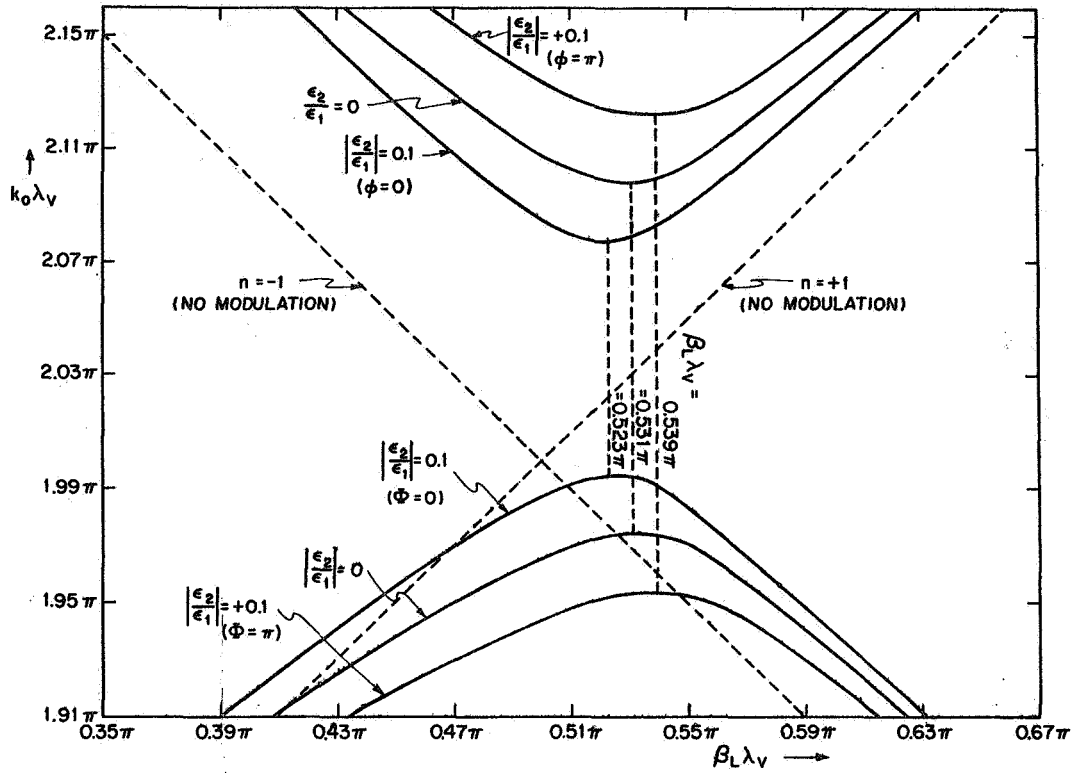


FIG. 1a. COUPLING BETWEEN $n=-1$ AND $n=+1$ HARMONICS; $\tau=0.25$; $\frac{\epsilon_1}{\epsilon_0} = 0.5$

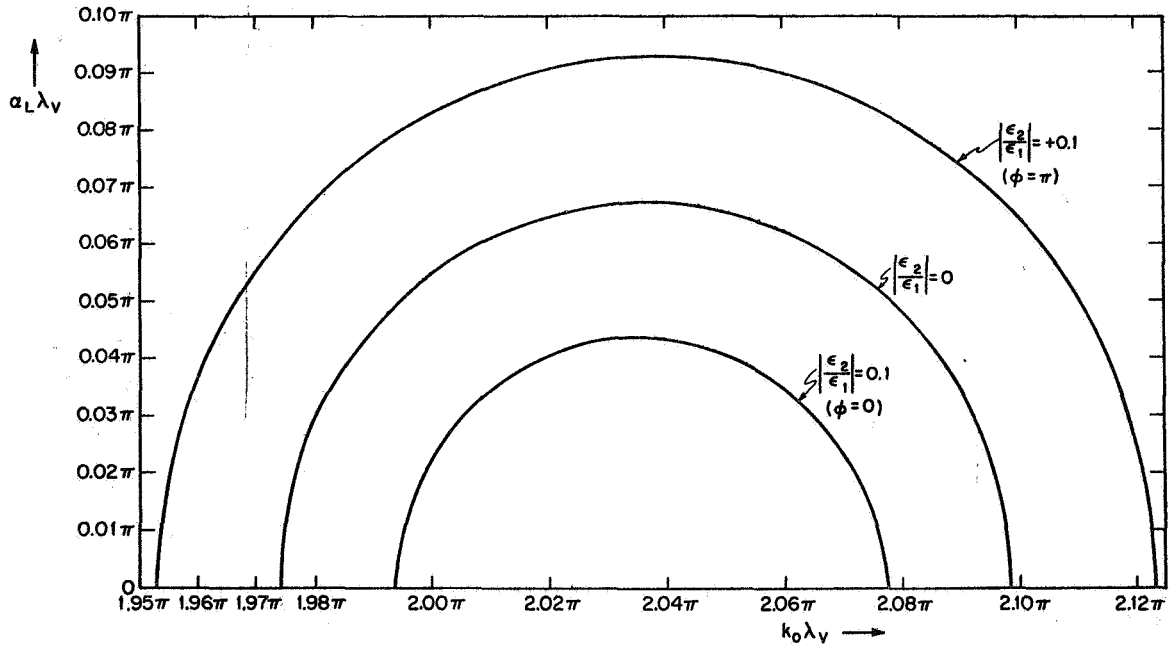


FIG. 1b. COUPLING BETWEEN $n=-1$ AND $n=+1$ HARMONICS; COMPLEX PROPAGATION CONSTANT; $\tau=0.25$; $\frac{\epsilon_1}{\epsilon_0} = 0.5$

semicircular manner. The complex propagation constant within the stop band is also significantly affected by the second harmonic pump wave. When $\epsilon_2 = 0$, the maximum value of the normalized complex propagation constant within the stop band is given by $0.531\pi + j0.067\pi$; with the second harmonic out of phase ($\phi = \pi$) and $|\epsilon_2/\epsilon_1| = +0.1$, the maximum value of the propagation constant in the stop band increases to $0.539\pi + j0.093\pi$ and reduces to $0.523\pi + j0.044\pi$ for the in-phase condition ($\phi = 0$, $|\epsilon_2/\epsilon_1| = +0.1$). From these computations it appears that the parametric coupling between the $n = +1$ and $n = -1$ space-time harmonics of the signal wave is stronger when the second harmonic of the pump wave is driven out of phase with the fundamental and decreases when it is in phase with the fundamental component of the pump wave.

Type 2. When the phase velocity of the pump wave is larger than the phase velocity of the signal wave in the unperturbed medium ($v > 1$), the parametric interactions are characterized by 'temporal instabilities' resulting in time growing oscillations. The nature of this oscillatory growth process has been well described by Cassedy [2]. The stop bands in the Brillouin diagram are inverted, wherein solutions for the dispersion relation are obtained only for complex frequencies and real wave numbers. The amplitude and phase of the second harmonic once again has a strong influence on both the width of the stop band in temporal frequency ($k_0 = \omega/c$) and on the growth rate of the oscillations; however, the effect of the phase relation of the second harmonic on the parametric growth rate is opposite to that observed in the Type 1 interaction where $v < 1$. Figs. 2a and b show the Brillouin diagram for the interaction between the $n = -1$ and $n = +1$ harmonics. When $\epsilon_2 = 0$, the inverted stop

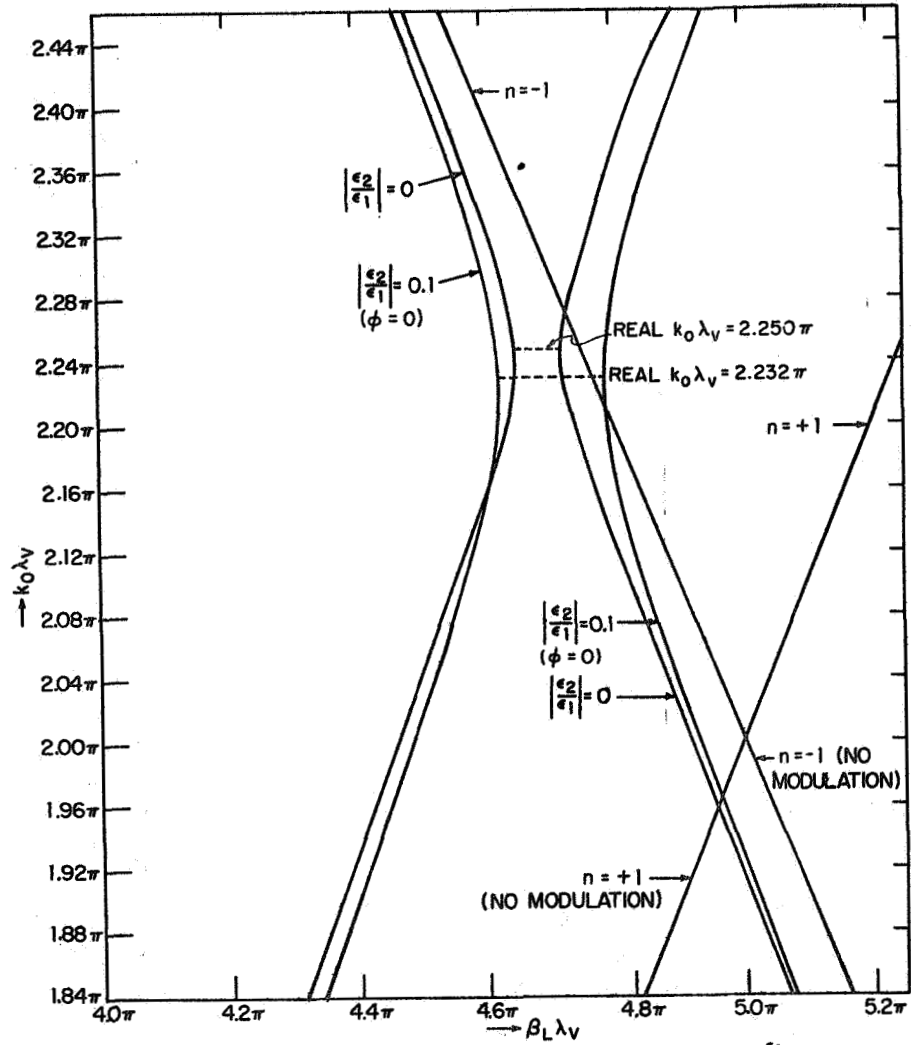


FIG. 2a. COUPLING BETWEEN $n = -1$ AND $n = +1$ HARMONICS; $\tau = 2.5$; $\frac{\epsilon_1}{\epsilon_0} = 0.5$.

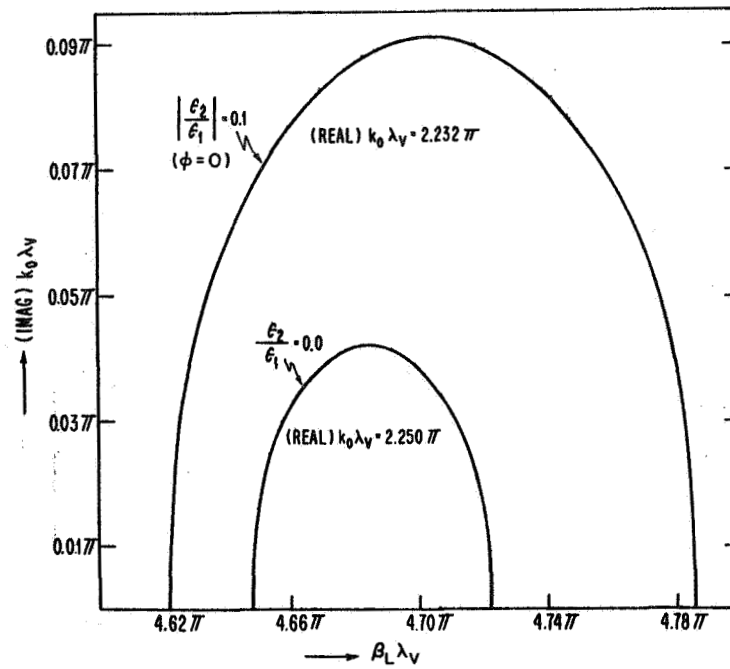


FIG. 2b. COUPLING BETWEEN $n = -1$ AND $n = +1$ HARMONICS; $\tau = 2.5$; $\frac{\epsilon_1}{\epsilon_0} = 0.5$

band extends from $\beta_L \lambda_v = 4.610\pi$ to 4.720π . When $\phi = 0$ (in phase) and $|\epsilon_2/\epsilon_1| = 0.1$, the stop band expands in width and covers the range $\beta_L \lambda_v = 4.625\pi$ to 4.785π . For fixed values of ϵ_1 , ϵ_2 , and ϕ , the temporal frequency within the stop band is constant but the growth rate (imaginary part of $k_0 \lambda_v$) varies in an approximately semicircular manner. The effect of the second harmonic pump wave on the growth rate is shown in Fig. 2b; when $\epsilon_2 = 0$, the maximum value of the complex $k_0 \lambda_v$ in the stop band is $2.250\pi + j0.042\pi$. When $|\epsilon_2/\epsilon_1| = +0.1$ and $\phi = 0$ (in phase condition), the maximum value of $k_0 \lambda_v$ is $2.232\pi + j0.09\pi$. The growth rate of the instability, therefore, increases with the amplitude of the second harmonic when it is in phase with the fundamental component of the pump wave.

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Acknowledgment

The author wishes to thank Professor R. W. P. King for his advice and encouragement and Mr. Marvin Siegel for his valuable help in formulating the computer program for solving the dispersion relation.